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On a paper of Lang and Maslamani

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Abstract

Let k be a field of characteristic 0, and let $f:k^n \to k^n$ be a polynomial map with components of the form $f_i = x_i + h_i$, where the h_i are monomials. If the Jacobian determinant of the map f is a nonzero constant, then f is a tame automorphism. If, in addition, each h_i is either constant or of degree 2 or more, then f is linearly triangularizable.

MSC: Primary: 14R15; secondary: 14E09

1. Introduction

Let k be a field of characteristic 0. Denote by $k[x_1,\ldots,x_n]$ the polynomial ring in the variables x_i, $i=1,\ldots,n$, with coefficients in k. Let $f:k^n \to k^n$ be a polynomial map with components $f_i \in k[x_1,\ldots,x_n]$ for $i=1,\ldots,n$. The Jacobian conjecture [4,1,2] asserts that if the determinant of the Jacobian matrix $J(f)$ of partials of f is a nonzero constant, then f is invertible, meaning that f is one-to-one and onto and that its inverse map is also polynomial. A special case was considered by Lang and Maslamani [5]. They proved

Theorem 1 (Lang and Maslamani [5]). *If $f_i = x_i + h_i$ for $i=1,\ldots,n$, with each h_i either 0 or a monomial of degree at least 2, and $\det(J(f))=1$, then f is invertible.*

A major portion of their paper uses rational weightings $w=(w_1,\ldots,w_n)\in\mathbb{Q}^n$, the associated degree function $\deg_w(x_1^{e_1}\cdots x_n^{e_n}) = w_1e_1+\cdots+w_ne_n$ on terms, and

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decompositions of the Jacobian determinant into w-homogeneous components, to prove by an ingenious argument that

Proposition 1 (Lang and Maslamani [5]). *If* $f_i = x_i g_i$ *for* $i = 1,\ldots,n$, *with* $g_i \in k[x_1,\ldots,x_n]$, *and* $\det(J(f)) \in k^* = k \setminus \{0\}$, *then each* $g_i \in k^*$.

They then use that result to prove Theorem 1. A refinement of Theorem 1 is proved in this paper, starting from Proposition 1. To explain it, consider the following example:

Example 1. Let $f : k^3 \to k^3$ be given by

$$(u, v, w) = f(x, y, z) = (x + y^2 z, y + z^2, z).$$

Then

$$J(f) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 2yz & y^2 \\ 0 & 0 & 2z \\ 0 & 0 & 0 \end{bmatrix},$$

represents $J(f)$ as $I + N$, with N strictly upper triangular. Thus $\det(J(f)) = 1$. One easily obtains the inverse of f by solving for z, y, and x, in that order. The result is

$$(x, y, z) = (u - w(v - w^2)^2, v - w^2, w) = (u - wv^2 + 2vw^3 - w^5, v - w^2, w).$$

It is easy to obtain the inverse of f in the example, because f is in (unit upper) triangular form; that is, $f_i = x_i + h_i$, where h_i depends only on the variables x_j for $j > i$. Such maps are tame automorphisms; that is, they are compositions of invertible linear maps and elementary automorphisms (an elementary automorphism f has all but one component of the form $f_i = x_i$ and the single exception is of the form $f_j = x_j + r$, where r is a polynomial that contains no occurrence of x_j). All this is not a coincidence.

Theorem 2. *Let* $f : k^n \to k^n$ *be a polynomial map over a field* k *of characteristic* 0, *with components* $f_i = x_i + h_i$, $i = 1,\ldots,n$, *where each* h_i *is a monomial. If* $\det J(f) \in k^*$, *then* f *is a tame automorphism* (*thus invertible*). *If, in addition, each* h_i *is constant or of degree* 2 *or more, then there is a permutation of the variables* $x_1,\ldots,x_n$ *which makes* f *unit upper triangular, so* f *is linearly triangularizable.*

Remark. The permutation is to be applied to both the variables and the components of f. That is, it is a linear change of variables that happens to be a permutation. If P is the associated permutation matrix, then the transformed map is $\tilde{f} := P'(f(Px))$, where P' is P-inverse $= P$-transpose. This is a special case of a linear triangularization $\tilde{f} := L^{-1}(f(Lx))$, in which L can be any invertible matrix of constants.

Remark. The statement of this theorem is intended to allow for cases in which some, or all, of the h_i are constant (possibly 0) or linear.

Corollary 1. *If* $f_i = a_i x_i + b_i + h_i$, *for* $i = 1, \ldots, n$, *with* $a_i \in k^*$, $b_i \in k$, *and each* h_i *is* 0 *or a monomial*, *then if* $\det(J(f)) \in k^*$, *it follows that* f *is invertible*.

The corollary follows immediately by subtracting the b_i and dividing by the a_i to obtain a map to which Theorem 2 can be applied.

The significance of the result obtained here is that while Theorems 1 and 2 guarantee invertibility for a significant class of maps in an arbitrary number of variables, they cannot produce any truly exotic examples of invertible polynomial maps, because they will all be tame. In fact, Theorem 1 considers only maps of the form $f(x) = x + h(x)$, where $x = (x_1, \ldots, x_n)$ and h consists of monomials that are "higher order terms" (that is, $h_i = 0$ or h_i is a monomial with deg $\geqslant 2$), and by Theorem 2 all possible examples can be trivially produced by choosing monomials to form a unit upper triangular map and then simply permuting the variables.

The proof is along the lines of the original proof of invertibility in [5] (specifically, it features analogues of Lemmas 3 and 7, and Proposition 8). There are some potential points of confusion in the original paper: (1) the monomials in question are sometimes, but not everywhere, required to be of degree greater than 1 (e.g., the abstract does not mention any restrictions on the degrees of the monomials), (2) early on an unnecessary assumption is made that k is algebraically closed, and (3) the terminology used in the proof of Proposition 8 is perhaps misleading and produces a somewhat cryptic proof. In fact it is easy to dismiss the very short proof of Proposition 8 as technically inadequate, but note that it touches on most of the ideas used in the proofs in this paper. The proof of Theorem 2 not only refines Theorem 1, but should help clarify (1)–(3).

2. Proof

Consider first a construction. Suppose that $f_i = x_i + h_i$, for $i = 1, \ldots, n$, and h_1 is a polynomial in $x_2, \ldots, x_n$ (that is, $\partial h_1/\partial x_1 = 0$). Let $J = J(f)$ be the Jacobian matrix of f. Assume $\det(J) \in k^*$. Column 1 of J contains the entries $1 = \partial f_1/\partial x_1, \partial f_2/\partial x_1, \ldots, \partial f_n/\partial x_1$. Now consider the matrix K obtained from J by subtracting multiples of the first column from each of the other columns as follows. For column j, with $j \geqslant 2$, subtract column 1 time $\partial h_1/\partial x_j$.

Then the entries at the top of columns 2 through n of K are zero

$$\partial f_1/\partial x_j - 1 * \partial h_1/\partial x_j = 0.$$

Further, the entry in position (i,j) of K for $i > 1, j > 1$ is

$$\partial f_i/\partial x_j - \partial f_i/\partial x_1 * \partial h_1/\partial x_j.$$

Let ϕ denote the substitution homomorphism from $k[x_1, \ldots, x_n]$ to $k[x_2, \ldots, x_n]$ that substitutes $-h_1$ for x_1. For $i \geqslant 2$, let $g_i = \phi(f_i) = f_i(-h_1, x_2, \ldots, x_n)$. Then the chain rules states that for $i > 1, j > 1$, ϕ applied to the (i,j) entry of K is exactly $\partial g_i/\partial x_j$.

So the submatrix of $\phi(K)$ obtained by deleting the first row and column is the Jacobian matrix of the $n-1$ variable map $g := (g_2, \ldots, g_n)$,

$$\phi(K) = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ \phi\left(\dfrac{\partial f_2}{\partial x_1}\right) & \dfrac{\partial g_2}{\partial x_2} & \ldots & \dfrac{\partial g_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \phi\left(\dfrac{\partial f_n}{\partial x_1}\right) & \dfrac{\partial g_n}{\partial x_2} & \ldots & \dfrac{\partial g_n}{\partial x_n} \end{bmatrix}.$$

It is clear that the determinant of the submatrix of K obtained by deleting row 1 and column 1 is the same as $\det(J)$. Since ϕ is k-linear the map g has properties similar to those of f. Specifically, for $i \geqslant 2$ one has $g_i = x_i + \phi(h_i)$ and $\det(J(g)) = \det(J(f))$.

An important point about this construction is that if h_1 is a monomial and a given h_i, $i > 1$ is a monomial, then $\phi(h_i)$ is also a monomial. Furthermore, if h_1 and h_i are both nonconstant monomials, the same is true for $\phi(h_i)$. In what follows, a monomial may be a constant, even zero.

Lemma 1. *If $f_i = x_i + h_i$, with each h_i a monomial, and $\det(J(f)) \in k^*$, then for $i = 1, \ldots, n$, either $f_i = a_i x_i, a_i \in k^*$ or $\partial h_i / \partial x_i = 0$.*

Proof. Permute the variables (and the components f_i correspondingly) so that there are (up to) four groups of components. First, take the components for which h_i is a nonconstant monomial that does not have x_i as a factor. Second, those f_i for which $h_i = x_i m_i$ for some nonconstant monomial $m_i \in k[x_1, \ldots, x_n]$. Third, those components with $h_i \in k^*$. Finally, the components for which $h_i = c_i x_i$ with $c_i \in k$; note that $a_i = 1 + c_i$ must be in k^*, for otherwise f would have component $f_i = a_i x_i = 0$, contradicting $\det(J(f)) \in k^*$. If there are no components of type 2 the proof is complete. So assume that there is at least one component of type 2. One may also assume that there are no components of type 1. To see this, suppose that there is a component of type 1. Assume it is f_1. Apply the construction described above, to obtain a map $g = (g_2, \ldots, g_n)$. Each component f_i of type 2,3, or 4 gives rise to a component g_i of g of the same type as f_i. One component of the first type is eliminated. Any other components of type 1 give rise to a component of type 1 or of type 2. So the number of components of type 2 never drops. Repeat until all components of the first type have been eliminated. At least one component is of type 2. Now consider Proposition 1 applied to the map obtained by subtracting off any constant terms, and observe that there is a contradiction. □

Lemma 1 states that there are no components of type 2. Call components of type 3 or 4 simple components ($f_i = x_i + b_i, b_i \in k^*$ or $f_i = a_i x_i, a_i \in k^*$). Call f simple if all of its components are simple. All simple components are affine (that is, linear in the x_i or linear in the x_i plus a constant term). There may be components of type 1 that are affine but not simple ($f_i = x_i + c_i x_j, c_i \in k^*, j \neq i$).

Lemma 2. *If $f_i = x_i + h_i$, with each h_i a monomial, all affine components of f are simple, and $\det(J(f)) \in k^*$, then either f is simple or it has a nonsimple component of the form $f_i = x_i + h_i$, with h_i in the k-subalgebra of $k[x_1, \dots, x_n]$ generated by the simple components of f.*

Proof. Assume f is not simple. Suppose there is a counterexample with $s > 0$ nonsimple components (placed first) and $n - s$ simple components. Note that the k-subalgebra generated by the simple components is just $A = k[x_{s+1}, \dots, x_n]$. If $s = 1$ (only a single nonsimple component), then $f_1 = x_1 + h_1$ and h_1 is a nonconstant monomial, of degree at least 2, that does not belong to $k[x_2, \dots, x_n]$ and hence is divisible by x_1; thus $h_1 = x_1 m_1$, where m_1 is not constant. Subtracting off any constant terms yields a map that contradicts Proposition 1. So suppose $s > 1$. By assumption, for $i = 1, \dots, s$, $h_i \notin A$ and so $\partial h_i / \partial x_i = 0$. In particular, $\partial h_1 / \partial x_1 = 0$. Now construct $g = (g_2, \dots, g_n)$ as before. One nonsimple component, the first, is eliminated. Any remaining nonsimple components give rise to nonaffine components (of type 1, since the previous lemma establishes that there will be no components of type 2). Such a component is of the form $g_j = x_j + \phi(h_j)$; h_j is a nonconstant monomial that does not contain x_j as a factor, and does not belong to A; it must therefore have a factor x_l, with $l \neq j, l \leqslant s$; $\phi(h_j)$ must contain the factor $\phi(x_l)$. Let A' be the k-subalgebra of $k[x_2, \dots, x_n]$ generated by the simple components of g. Since the distinction between simple and nonsimple components is preserved in passing from f to g, it follows that $A' = \phi(A) = k[x_{s+1}, \dots, x_n]$ with the usual identification of $k[x_2, \dots, x_n]$ as a k-subalgebra of $k[x_1, \dots, x_n]$. If $l \neq 1$, it is clear that $\phi(h_j) \notin A'$. If $l = 1$, then that factor gets replaced by $-h_1$, which itself must contain a factor x_m, with $m \neq 1, m \leqslant s$. Again, $\phi(h_j) \notin A'$. So g is a counterexample with $s - 1$ nonsimple components. Repeat until the resulting map has only a single nonsimple component. Contradiction. □

Corollary 2. *If $f_i = x_i + h_i$, with each h_i a monomial, and $\det(J(f)) \in k^*$, then f contains at least one affine component.*

Proof. Suppose not. Then all affine components of f are simple, and the k-subalgebra generated by the simple components of f is just k. By Lemma 2, f has a component of the form $f_i = x_i + b_i$, where $b_i \in k$. That is, it has a simple component. Contradiction. □

Recall that a principal submatrix of a square matrix is obtained by deleting a set of rows and the columns with the same indices. A principal minor is the determinant of a principal submatrix.

Proposition 2. *If $f_i = x_i + h_i$, with each h_i a monomial, all affine components of f are simple, and $\det(J(f)) \in k^*$, then f is invertible and all of the principal minors of $J(f)$ are nonzero constants.*

Proof. Permuting the variables (and the components in the same way) does not change the set of principal minors of the Jacobian matrix, provided they are all constant for at least one (hence both) of the two matrices. That follows from the chain rule and the invariance of (appropriately matched) principal minors under a permutation similarity. The equality $k[f_1,\ldots,f_n]=k[x_1,\ldots,x_n]$ is a necessary and sufficient condition for invertibility of f (since it states that the x_i are polynomials in $f_1,\ldots,f_n$). If f is simple, then f is clearly invertible and its Jacobian matrix is a constant diagonal matrix, hence all of the principal minors of $J(f)$ are constant. Now suppose that f has $s>0$ nonsimple components and $n-s$ simple components. Place the nonsimple components first. Let A be the k-subalgebra generated by the simple components. Then $A=k[x_{s+1},\ldots,x_n]$ (or just k if $s=n$). Clearly $k[f_1,\ldots,f_n]=k[f_1,\ldots,f_s,x_{s+1},\ldots,x_n]$. By Lemma 2, at least one of the first s components is of the form $f_i=x_i+h_i$ with $h_i\in A$. Without loss of generality, assume it is f_s.

Then

$$J(f)=\begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_{s-1}} & \dfrac{\partial f_1}{\partial x_s} & \dfrac{\partial f_1}{\partial x_{s+1}} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \dfrac{\partial f_{s-1}}{\partial x_1} & \cdots & \dfrac{\partial f_{s-1}}{\partial x_{s-1}} & \dfrac{\partial f_{s-1}}{\partial x_s} & \dfrac{\partial f_{s-1}}{\partial x_{s+1}} & \cdots & \dfrac{\partial f_{s-1}}{\partial x_n} \\ 0 & \cdots & 0 & 1 & \dfrac{\partial h_s}{\partial x_{s+1}} & \cdots & \dfrac{\partial h_s}{\partial x_n} \\ 0 & \cdots & 0 & 0 & \Delta_{s+1} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & \Delta_n \end{bmatrix},$$

where Δ_i, $i=s+1,\ldots,n$ are the constants along the diagonal of the diagonal matrix $J(f_{s+1},\ldots,f_n)$ ($\Delta_i=1$ if $f_i=x_i+b_i$; $\Delta_i=a_i$ if $f_i=a_ix_i$). Since $f_s=x_s+h_s$ and $h_s\in k[x_{s+1},\ldots,x_n]$, it follows that $k[f_1,\ldots,f_n]=k[f_1,\ldots,f_s,x_{s+1},\ldots,x_n]=k[f_1,\ldots,f_{s-1},x_s,\ldots,x_n]$. Thus if f_s is replaced by x_s, the resulting map has one more simple component and the k-subalgebra generated by the components is unchanged. Furthermore, the set of principal minors of the Jacobian matrix remains the same, since the new Jacobian is the same as the old, except that the terms $\partial h_s/\partial x_i$, $i=s+1,\ldots,n$ have been zeroed out. Repeat this process until f is simple. The chain of equalities of subalgebras establishes $k[f_1,\ldots,f_n]=k[x_1,\ldots,x_n]$, so (the original) f is invertible. Furthermore, the final map has nonzero constant principal minors, so that working back through any permutations made shows that the original map f has nonzero constant principal minors. □

Proposition 3. *If $f_i=x_i+h_i$, with each h_i a monomial, all affine components of f are simple, and* $\det(J(f))\in k^*$, *then f is a tame automorphism.*

Proof. By Proposition 2, all the leading principal minors (the determinants of the submatrices defined by the first r rows and columns of $J(f)$, $r=1,\ldots,n$) are nonzero constants. But that is the definition of a Samuelson map. And Samuelson maps are tame [3,2, Section 10.1, Exercise 1, p. 244]. □

Now consider the case in which each h_i is either constant or a monomial of degree at least 2. Then $J(f)$ evaluated at $x_1 = x_2 = \cdots = x_n = 0$ is the identity matrix I. All the principal minors of I are 1, and since the principal minors of $J(f)$ are constant, they must be the same as those of I, and so they are all 1.

Lemma 3. *Let M be a square matrix of constants. Assume all the principal minors of M are* 1. *Then $M = I + N$, where all of the principal minors of N are* 0.

Proof. Note that for any $n \times n$ matrix N,

$$\mathrm{d}/\mathrm{d}t \det(N + tI) = \sum_{i=1}^{n} \det(N_i + tI), \tag{1}$$

where N_i is the matrix obtained from M by deleting row and column number i, i.e., $\mathrm{d}/\mathrm{d}t \ \det(N + tI)$ is the sum of the principal minors of size $n - 1$ of $N + tI$. By assumption all the principal minors of $N + I$ are 1. It follows by induction that all $m \times m$ principal minors of $N + tI$, $m = 1, \ldots, n$, are equal to t^m. For $m = 1$ this is trivial. For $m > 1$, let $\det(L + tI)$ be a principal minor of $N + tI$ of size $m \times m$. By (1) and induction, $\mathrm{d}/\mathrm{d}t \ \det(L + tI) = mt^{m-1}$, and so $\det(L + tI)$ must be t^m, since $\det(L + I) = 1$. Take $t = 0$ to show that all the principal minors of N are 0. □

Remark. Thanks to Robert B. Israel for suggesting this proof. The result also follows easily from the usual characterization of the coefficients of the characteristic polynomial as sums of principal minors.

Proposition 4. *If $f_i = x_i + h_i$, $\det(J(f)) \in k^*$, and each h_i is a constant or a monomial of degree at least* 2, *then there is a permutation of the variables and components that puts f in unit upper triangular form.*

Proof. $J(f) = I + J(h)$. Apply the preceding lemma pointwise at every point of k^n, and conclude that all the principal minors of $J(h)(p)$ are 0 for every $p \in k^n$. Since k is infinite and the principal minors of $J(h)$ are polynomials, they must all be the constant 0. Consider $N = J(h)$ as a matrix of size $n \times n$ over $k[x_1, \ldots, x_n]$ (or over the rational function field $k(x_1, \ldots, x_n)$). By van den Essen [2, Proposition 6.3.9, p. 128] there is a permutation matrix P such that $P^{-1}NP$ is upper triangular with zeroes on the diagonal. Then it follows by the chain rule that $P^{-1}f(Px)$ is in unit upper triangular form (this is also the exact point of [2, Proposition 6.3.12, p. 129]). □

Next, consider maps with nonsimple affine components for which the affine components form a closed system as follows: call a map affine closed if every affine component contains only variables x_j for which f_j is affine. If $f_i = x_i + h_i$ for monomials h_i, $i = 1, \ldots, n$, this means that any nonsimple affine component is of the form $f_i = x_i + c_i x_j$ with f_j also an affine (possibly simple) component.

Lemma 4. *If $f_i = x_i + h_i$, with each h_i a monomial, and f is affine-closed, and* $\det(J(f)) \in k^*$, *then f is a tame automorphism.*

Proof. Permute the variables (and the components f_i correspondingly) to place the nonaffine components first, followed by the affine components. This does not affect the hypotheses or the conclusion. Let s be the number of nonaffine components. Write $J(f)$ as a partitioned matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$, where A is a block of size $s \times s$. The fact that f is affine closed means that $C = 0$. Furthermore, D is a matrix of constants. Since $\det(D)$ divides $\det(J(f))$, D is an invertible matrix. Consider Lf, where L is the invertible linear map with block diagonal matrix $\begin{bmatrix} I & 0 \\ 0 & D^{-1} \end{bmatrix}$. The first s components of Lf are just $f_1, \dots, f_s$. The final $n - s$ components of Lf are all of the form $x_i + b_i$, where $b_i \in k$. By Proposition 3, Lf is tame, hence so is f. □

Proposition 5. *If $f_i = x_i + h_i$, with each h_i a monomial, and* $\det(J(f)) \in k^*$, *then f is a tame automorphism.*

Proof. By induction on the number r of affine components. By Corollary 2, $r \geqslant 1$. If f is affine closed, then f is tame by Lemma 4. If not, there is a component of the form $f_i = x_i + c_i x_j$, where f_j is not affine. Let $f_j = x_j + h_j$, with $\deg(h_j) \geqslant 2$. Replace f_i by $f_i^* = x_i + c_i x_j - c_i(x_j + h_j) = x_i - c_i h_j$, to yield a new map f^*. Then $f^* = Ef$, where E is the elementary matrix with $E_{ii} = 1$, $i = 1, \dots, n$ and a single nonzero off diagonal entry $E_{ij} = -c_i$. Clearly f^* satisfies the hypotheses, but r is one less. By induction, f^* is tame, and hence so is f. □

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